

# On Unique Games with Negative Weights

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**Abstract.** In this paper, the author defines Generalized Unique Game Problem (GUGP), where weights of the edges are allowed to be negative. Focuses are made on two special types of GUGP, GUGP-NWA, where the weights of all edges are negative, and GUGP-PWT( $\rho$ ), where the total weight of all edges are positive and the negative/positive ratio is at most  $\rho$ . The author investigates the counterparts of the Unique Game Conjecture for the two types of generalized unique game problems. The author proves Unique Game Conjecture holds false on GUGP-NWA by giving a factor 2 approximation algorithm for Max GUGP-NWA, and Unique Game Conjecture holds true on GUGP-PWT(1) by reducing the parallel repetition of Max 3-Cut Problem to GUGP-PWT(1). The author poses an open problem whether Unique Game Conjecture holds true on GUGP-PWT( $\rho$ ) with  $0 < \rho < 1$ .

## 1 Introduction

The Unique Game Conjecture is put forward by Khot on STOC 2002 as a powerful tool to prove lower bound of inapproximability for combinatorial optimization problems[1]. It has been shown by researchers a positive resolution of this conjecture would imply improved even best possible hardness results for many famous problems, to name a few, Max Cut, Vertex Cover, Multicut, Min 2CNF Deletion, making an important challenge to prove or refute the conjecture.

Some variations of the Unique Game Conjecture have been mentioned. Rao proves a strong parallel repetition theorem which shows Weak Unique Game Conjecture is equivalent to the Unique Game Conjecture[2]. Khot et al. show that the Unique Game Conjecture on Max 2LIN(q) is equivalent to Unique Game Conjecture[3]. Arora et al. show the Unique Game Conjecture cannot hold when the constrain graph is an expander beyond a certain threshold, and put forward the Unique Game Conjecture with Expansion[4]. Bansal et al. use the Unique Game Conjecture with Vertex Expansion in [5].

Khot pose the d-to-1 Conjectures in his original paper[1] for  $d \geq 2$ . O'Donnell et al. show a tight hardness for approximating satisfiable constraint satisfaction problem on 3 Boolean variables assuming the d-to-1 Conjecture for any fixed  $d$ [6]. Dinur et al. use the 2-to-2 Conjecture and the  $\bowtie$  conjecture to derive the hardness results of Approximate Coloring Problem, and prove that Khot's d-to-1 Conjecture implies their d-to-d Conjecture[7]. Guruswami et al. use the 2-to-1 Conjecture to derive the hardness result of Maximum k-Colorable Subgraph Problem[8]. It is unknown whether the Unique Game Conjecture implies any of the d-to-1 Conjectures, or vice versa.

In this paper, the author relaxes Unique Game Problem (UGP) by defining Generalized Unique Game Problem (GUGP), where weights of the edges are allowed to be negative. Focuses are made on two special types of GUGP, GUGP-NWA, where the weights of all edges are negative, and GUGP-PWT( $\rho$ ), where the total weight of all edges are positive and the negative/positive ratio is at most  $\rho$ . Max GUGP-NWA can be restated as 2-Prover 1-Round Problem where each relation is a complement of a permutation. GUGP-PWT( $\rho$ ) over  $1 \geq \rho \geq 1$  forms a phase transition from a hard  $(1 - \zeta, \delta)$ -gap problem of 2-Prover 1-Round Problem to Min UGP.

The author investigates the counterparts of the Unique Game Conjecture for the two types of generalized unique game problems. The author proves Unique Game Conjecture holds false on GUGP-NWA by giving a factor 2 approximation algorithm for Max GUGP-NWA, and Unique Game Conjecture holds true on GUGP-PWT(1) by reducing the parallel repetition of Max 3-Cut Problem to GUGP-PWT(1). It is shown the  $(1 - \zeta, \delta)$ -gap hardness of problems in GUGP-PWT( $\rho$ ) possesses the compactness property when  $\rho \rightarrow 0$ . The author poses an open problem whether Unique Game Conjecture holds true on GUGP-PWT( $\rho$ ) with  $0 < \rho < 1$ .

Section 2 demonstrates some definitions and conjectures. The author shows the main results for GUGP-NWA in Section 3 and for GUGP-PWT( $\rho$ ) in Section 4. Section 5 is some discussions.

## 2 Preliminaries

In 2-Prover 1-Round Game Problem, we are given an graph  $G = (V, E)$ , with each edge  $e$  having a weight  $w_e \in \mathbb{Q}^+$ . We are also given a set of  $k$  labels, which we identify with  $[k] = \{1, \dots, k\}$ . Each edge  $e = (u, v)$  in the graph comes equipped with a relation  $R \subseteq [k] \times [k]$ . The solution of the problem is a labeling  $f : V \rightarrow [k]$  that assigns a label to each vertex of  $G$ . An edge  $e = (u, v)$  is said to be satisfied under  $f$  if  $(f(u), f(v)) \in R$ , else is said to be unsatisfied. The object of the problem is to find a labeling maximizing the total weight of the satisfied edges.

Unique Game Problem (UGP) is a special type of 2-Prover 1-Round Game Problem. In UGP, we are given an graph  $G = (V, E)$ , a weight function  $w_e \in \mathbb{Q}^+$  for  $e \in E$ , and a set of labels,  $[k]$ . Each edge  $e = (u, v)$  in the graph comes equipped with a permutation  $\pi_e : [k] \rightarrow [k]$ . The solution of the problem is a labeling  $f : V \rightarrow [k]$  that assigns a label to each vertex of  $G$ . An edge  $e = (u, v)$  is said to be satisfied under  $f$  if  $\pi_e(f(u)) = f(v)$ , else is said to be unsatisfied. Note that we allow  $G$  is a graph with parallel edges, i.e., there exist more than one edges between two vertices.

It is possible to define two optimization problems in this situation. In Max UGP, the value of the instance is defined as the maximum fraction of the total weight of the satisfied edges. In Min UGP, the value of the instance is defined as the minimum fraction of the total weight of the unsatisfied edges. The total weight of all edges is normalized to 1.

Khot initiates much of the interest in the following conjecture by showing that many hardness results stem from it. It basically states that it is NP hard to distinguish whether many or only few edges are satisfied.

**Conjecture 1. ([1] Unique Game Conjecture)** *For every  $\zeta, \delta > 0$ , there is a  $k = k(\zeta, \delta)$  such that it is NP hard to distinguish whether an instance of Max UGP with  $k$  labels has a value more than  $1 - \zeta$  or less than  $\delta$ .*

The conjecture can be restated in Min UGP form, and the two conjectures are equivalent.

**Conjecture 2. (Unique Game Conjecture in Min UGP Form)** *For every  $\zeta, \delta > 0$ , there is a  $k = k(\zeta, \delta)$  such that it is NP hard to distinguish whether an instance of Min UGP with  $k$  labels has a value less than  $\zeta$  or more than  $1 - \delta$ .*

In this paper, the author defines Generalized Unique Game Problem (GUGP), where weights of the edges are allowed to be negative. In GUGP, we are given an graph  $G = (V, E)$  possibly having parallel edges, weight function  $w_e \in \mathbb{Q}$  for  $e \in E$ , and the set of labels,  $[k]$ . Each edge  $e = (u, v)$  in the graph comes equipped with a permutation  $\pi_e : [k] \rightarrow [k]$ .

Note that  $w_e$  could be positive or negative. We assume there are no edges with zero weight for sake of clearance. Let  $\Sigma_G$  be the total of the positive weights of edges,  $W_G^-$  be the total of the negative weights of edges, and  $\Sigma_G = W_G^+ + W_G^-$  be the total weight of all edges. We call  $r_G = |W_G^-|/W_G^+$  the negative/positive ratio of the instance.

The solution of GUGP is a labeling  $f : V \rightarrow [k]$  that assigns a label to each vertex of  $G$ . It is also possible to define two optimization problems. In Max GUGP, the goal is to maximize the total weight of the satisfied edges. In Min GUGP, the goal is to minimize the total weight of the unsatisfied edges. GUGP-NWA and GUGP-PWT are two special types of GUGP.

In GUGP-NWA, the weight of all edges are negative. In Max GUGP-NWA, we seek to minimize the total weight of the unsatisfied edges, i.e. to maximize the *absolute value* of the total weight of the unsatisfied edges. The value of Max GUGP-NWA is defined as maximum fraction of the absolute value of the total weight of the unsatisfied edges, in condition  $|W_G^-|$  is normalized to 1. In Min GUGP-NWA, we seek to maximize the total weight of the satisfied edges, i.e. to minimize the *absolute value* of the total weight of the satisfied edges. The value of Min GUGP-NWA is defined as minimum fraction of the absolute value of the total weight of the satisfied edges, in condition  $|W_G^-|$  is normalized to 1.

We give the counterpart of the Unique Game Conjecture on GUGP-NWA as follows:

**Conjecture 3. (Unique Game Conjecture on GUGP-NWA)** *For every  $\zeta, \delta > 0$ , there is a  $k = k(\zeta, \delta)$  such that it is NP hard to distinguish whether an instance of Max GUGP-NWA with  $k$  labels has a value more than  $1 - \zeta$  or less than  $\delta$ .*

In GUGP-PWT, the total weight of all edges is positive. In GUGP-PWT, we seek to minimize the total weight of the unsatisfied edges. The value of Min GUGP-PWT is defined as the total weight of the unsatisfied edges divided by  $\Sigma_G$ . Note that the value of GUGP-PWT could be negative or more than 1. In an instance  $G$  of GUGP-PWT,

let  $W_G(f)$  be the total weight of the unsatisfied edges under labeling  $f$ , let the optimal labeling be  $f^*$ . The value of the instance is  $Val(G) = W_G(f^*)/\Sigma_G$ . We define GUGP-PWT( $\rho$ ) as the subproblem of GUGP-PWT where the negative/positive ratio is at most  $\rho$ , where  $\rho$  is a constant. Since the negative/positive ratio is always less than 1, we set the range of  $\rho$  to be  $0 < \rho \leq 1$ . Note that GUGP-PWT(0) is Min UGP itself.

We give the counterpart of the Unique Game Conjecture on GUGP-PWT( $\rho$ ) as follows:

**Conjecture 4. (Unique Game Conjecture on GUGP-PWT( $\rho$ ))** *For every  $\zeta, \delta > 0$ , there is a  $k = k(\zeta, \delta)$  such that it is NP hard to distinguish whether an instance of GUGP-PWT( $\rho$ ) with  $k$  labels has a value less than  $\zeta$  or more than  $1 - \delta$ .*

### 3 GUGP-NWA

In this section, we prove Min GUGP-NWA is NPO-complete, i.e. it is NP-hard to approximate Min GUGP-NWA within any factor of  $poly(n)$ , and we prove Max GUGP-NWA can be approximated with factor 2, and Conjecture 3 is refuted as a corollary.

**Theorem 1.** *Min GUGP-NWA is NPO-complete.*

*Proof.* Min GUGP-NWA can be restated as: In the situation of the unique game problem, the goal is to find minimum fraction of the total weight of the *satisfied edges*. We construct an approximation ratio preservation reduction from TSP to the above problem.

Given an instance of TSP problem  $G = (V, E)$ , where each edge of  $E$  has a weight  $w_e \in \mathbb{Q}^+$ . Denote  $n = |V|$ . The instance of Min GUGP-PWT is a graph  $G' = G'(V, E')$ , with each edge  $e' \in E'$  having a weight  $w'(e')$ , and with the labeling set  $[n]$ . For each edge  $e = (u, v) \in E$ , there are three parallel edges  $e^-, e^+$  and  $e^-$  between  $u$  and  $v$  in  $E'$ .  $e^-$  has weight  $M$  and equipped with permutation  $\pi^- = \{(1, 1), (2, 2), \dots, (n, n)\}$ . Let  $M = n \cdot \text{Max}(w)$ , where  $\text{Max}(w)$  is the maximum weight of all edges in  $G$ .  $e^+$  has weight  $w(e)$  and equipped with permutation  $\pi^+ = \{(1, 2), (2, 3), \dots, (n, 1)\}$ .  $e^-$  has weight  $w(e)$  and equipped with permutation  $\pi^- = \{(1, n), (2, 1), \dots, (n, n-1)\}$ .

Given a solution of TSP problem, a Hamiltonian cycle  $C$ , we can assign label 1 to  $n$  to vertices of  $C$  along  $C$  in  $G'$ , and the total weight of satisfied edges in  $G'$  is exactly the total weight of edges on  $C$  in  $G$ .

In the other direction, given a labeling  $f$  of  $G'$ , if there are two vertices assigned with the same label, the total weight of the satisfied edges is at least  $M$ . Otherwise all vertices are assigned with label from 1 to  $n$  respectively, let  $u_i$  be the vertices assigned label  $i$  for  $1 \leq i \leq n$ , and  $e_i^+ \in E'$  be the edge between  $u_i$  and  $u_{i+1 \bmod n}$  equipped with permutation  $\pi^+$ . The total weight of the satisfied edges is equal to  $\sum_{1 \leq i \leq n} w'(e_i^+)$ . Let  $C$  be the Hamiltonian cycle of  $G$  which consists of vertices from  $u_1$  to  $u_n$ , then the total weight of  $C$  in  $G$  is exactly the total weight of satisfied edges under  $f$  in  $G'$ .  $\square$

**Theorem 2.** *Max GUGP-NWA can be approximated within factor 2.*

*Proof.* Max GUGP-NWA can be restated as the following 2-prover 1-round game problem. We are given an graph  $G = (V, E)$ , a weight function  $w_e \in \mathbb{Q}^+$ , and the set of labels,  $[k]$ . Each edge  $e = (u, v)$  in the graph comes equipped with a relation  $\bar{\pi}_e = [k] \times [k] - \pi_e$ , where  $\pi_e : [k] \rightarrow [k]$  is a permutation. The solution of the problem is a labeling  $f : V \rightarrow [k]$  that assigns a label to each vertex of  $G$ . An edge  $e = (u, v)$  is said to be satisfied under  $f$  if  $(f(u), f(v)) \in \bar{\pi}_e$ . The value of the instance is defined as the maximum fraction of the total weight of the satisfied edges

We describe an approximation algorithm that finds a solution under which the fraction of the total weight of the satisfied edges is at least  $1/2$ , which is at least half of the value of the instance.

In the beginning of the algorithm, assign arbitrary labels to all vertices. Let  $Ass(v, e)$  be the predicate whether  $v \in V$  is associated with  $e \in E$ , and  $Sat(e)$  be the predicate whether  $e$  is satisfied by current labeling. In each iteration of the algorithm, let  $U^< = \{v \in V \mid \sum_{Ass(v,e) \wedge Sat(e)} w_e < \frac{1}{2} \sum_{Ass(v,e)} w_e\}$ , and  $U^{\geq} = \{v \in V \mid \sum_{Ass(v,e) \wedge Sat(e)} w_e \geq \frac{1}{2} \sum_{Ass(v,e)} w_e\}$ . If  $U^< = \emptyset$ , the algorithm stops. Otherwise, choose a vertex  $u$  from  $U^<$ , suppose the label assigned to  $u$  is  $f_1$ , assign another label  $f_2$  to the vertex. If an edge  $e = (u, v)$  is unsatisfied under the old labeling, it must be the case  $(f_1, f(v)) \notin \bar{\pi}_e$ , and  $\pi_e(f_1) = f(v)$ . So  $\pi_e(f_2) \neq f(v)$ , and  $(f_2, f(v)) \in \bar{\pi}_e$ . Therefore, in the new labeling vertex  $u$  satisfies the condition of  $U^{\geq}$ , and we move it from  $U^<$  to  $U^{\geq}$ . Since after each iteration, the number of vertices in  $W$  is increasing by 1, the algorithm stops in  $|V|$  iterations.  $\square$

**Corollary 1.** *Conjecture 3 holds false.*

## 4 GUGP-PWT( $\rho$ )

### 4.1 Parallel Repetition of Max 3-Cut

In Max 3-Cut Problem, the instance is a graph  $G = (V, E)$ , the value of the instance is the maximum fraction of properly colored edges of  $G$  under a 3-coloring. 3-coloring of a graph in a color set  $[3]$  is a function  $\chi : V \rightarrow [3]$ . We say an edge is properly colored under a 3-coloring if its endpoints receive distinct colors. A graph is 3-colorable if there is a coloring under which all edges are proper colored. Max 3-Cut Problem can be viewed as a 2-prover 1-round game with  $k = 3$ , where each edge comes equipped with a relation  $\pi_e = \{(1, 2), (2, 3), (3, 1), (2, 1), (3, 2), (1, 3)\}$ .

Given an instance of Max 3-Cut Problem,  $G$ , we define the  $l$ -fold parallel repetition of the instance,  $G^l = G^l(V^l, E^l)$ , as follows.  $G^l = \{< u_1, \dots, u_l > \mid u_i \in V, 1 \leq i \leq l\}$ , and  $E^l = \{(< u_1, \dots, u_l >, < v_1, \dots, v_l >) \mid (u_i, v_i) \in E, 1 \leq i \leq l\}$ . The value of the instance is the maximum fraction of properly colored edges under a  $l$ -fold 3-coloring. Define a  $l$ -fold 3-coloring of  $G^l$  in the color set  $[3]^l$  be the function  $\chi^l : V \rightarrow [3]^l$ , and let  $m(\chi^l)$  be the number of properly colored edges under  $\chi^l$ . We say an edge  $e = (< u_1, \dots, u_l >, < v_1, \dots, v_l >)$  in  $E^l$  is properly colored under a  $l$ -fold 3-coloring if  $\chi(u_i) \neq \chi(v_i)$  for any  $1 \leq i \leq l$ . The graph  $G^l$  is 3-colorable if there is a  $l$ -fold 3-coloring under which all edges are proper colored.

Petrant [9] shows that Max 3-Cut Problem possesses a hard gap at location 1, i.e. it is NP-hard to distinguish whether the instance is 3-colorable or whether has value

at most  $1 - \gamma$  for some constant  $\gamma$ . The constant is presumably very small and not determined in his paper. V. Guruswami et al. [8] make the constant clear to be  $1/33 - \alpha$  for any  $\alpha > 0$ .

**Lemma 1.** ([9] Theorem 3.3) *It is NP-hard to distinguish whether the instance of Max 3-Cut Problem is whether 3-colorable or has value at most  $1 - \gamma$  for some constant  $\gamma > 0$ .*

Raz's Parallel Repetition Theorem is used to enlarge the gap of 2-prover 1-round game with perfect completeness. We introduce Lemma 2 when applying Parallel Repetition Theorem to  $l$ -fold parallel repetition of Max 3-Cut Problem, and get Lemma 3 by a gap-reduction.

**Lemma 2.** ([10] Theorem 1.1) *If an instance of Max 3-Cut Problem has value  $1 - \gamma$ , the value of the  $l$ -fold parallel repetition of the instance is at most  $(1 - \gamma^{c_1})^{c_2 l}$ , where  $c_1$  and  $c_2$  are two positive constants.*

**Lemma 3.** *For any constant  $\delta > 0$ , there is a constant  $l = l(\delta)$  such that it is NP-hard to distinguish whether the instance of  $l$ -fold parallel repetition of Max 3-Cut Problem is  $l$ -fold 3-colorable or has value at most  $\delta$ .*

*Proof.* Given an instance of Max 3-Cut Problem,  $G = (V, E)$ , let  $l$  be the integer no less than  $\frac{\ln \delta}{c_2 \ln(1 - \gamma^{c_1})}$ . Let  $G^l$  be the  $l$ -fold parallel repetition of  $G$ . The proof can be achieved by the following two steps and Lemma 2.

**Completeness.** Suppose  $G$  is 3-colorable. Define a  $l$ -fold 3-coloring of  $G^l$  as  $\chi^l(< v_1, \dots, v_l >) = < \chi^*(v_1), \dots, \chi^*(v_l) >$ , where  $\chi^*$  is the optimal 3-coloring of  $G$ . Since all edges in  $G$  are properly colored under  $\chi^*$ , all edges in  $G^l$  are properly colored under  $\chi^l$ . Therefore,  $G^l$  is  $l$ -fold 3-colorable.

**Soundness.** Suppose  $G$  has value at most  $1 - \gamma$ . By Lemma 2, the value of  $G^l$  is at most  $(1 - \gamma^{c_1})^{c_2 l}$ , which is at most  $\delta$  by the definition of  $l$ .  $\square$

## 4.2 Unique Game Conjecture on GUGP-PWT( $\rho$ )

Let us show Conjecture 4 holds true at the boundary of the range of  $\rho$ .

**Theorem 3.** *Conjecture 4 holds true for  $\rho = 1$ .*

*Proof.* Given two constants  $\zeta, \delta > 0$ , let  $G^l = (V^l, E^l)$  be an instance of the parallel repetition of the Max 3-Cut Problem. The instance of Min GUGP-PWT(1) is a graph  $G' = (V^l, E')$ , with labeling set  $[3^l]$ .

To accomplish the reduction, we design a gadget as replacing each edge  $e = (u, v)$  in  $E^l$  with  $3^l$  parallel edges  $e_{i_1, \dots, i_l}$  for  $1 \leq i_j \leq 3, 1 \leq j \leq l$  between  $u$  and  $v$  in  $E'$ . Let  $E^=$  be the set of edges such that at least one index is 1, and  $E^\neq$  be the set of edges such that all indexes are 2 or 3. Note that  $|E^=| = 3^l - 2^l$  and  $|E^\neq| = 2^l$ . Edges in  $E^=$  has weight  $w_x$ , and edges in  $E^\neq$  has weight  $w_y$ .  $e_{i_1, \dots, i_l}$  for  $1 \leq i_j \leq 3, 1 \leq j \leq l$  is equipped with

permutation  $\pi_{i_1, \dots, i_l} = \{(\langle f_1, \dots, f_l \rangle, \langle f_1 + i_1 - 1 \bmod 3, \dots, f_l + i_l - 1 \bmod 3 \rangle) | f_j \in [3], 1 \leq j \leq l\}$ .

Suppose two vertices  $u$  and  $v$  are assigned labels  $\langle f_{u,1}, \dots, f_{u,l} \rangle$  and  $\langle f_{v,1}, \dots, f_{v,l} \rangle$  respectively. We require: (i) when  $f_{u,j} = f_{v,j}$  for at least one  $1 \leq j \leq l$ , the total weight of the unsatisfied edges between  $u$  and  $v$  in  $E'$  is 1; (ii) when  $f_{u,j} \neq f_{v,j}$  for any  $1 \leq j \leq l$ , the total weight of the unsatisfied edges between  $u$  and  $v$  in  $E'$  is 0.

Let us determine the value of  $w_x$  and  $w_y$ . By the linear equations,

$$\begin{cases} (3^l - 2^l - 1)w_x + 2^l w_y = 1 \\ (3^l - 2^l)w_x + (2^l - 1)w_y = 0 \end{cases}$$

we have  $w_x = -\frac{2^l - 1}{3^l - 1}$ , and  $w_y = \frac{3^l - 2^l}{3^l - 1}$ . Note that  $w_x < 0$  and  $w_y > 0$ .  $W_{G'}^+ = \frac{2^l(3^l - 2^l)}{3^l - 1}|E^l|$ ,  $W_{G'}^- = -\frac{(2^l - 1)(3^l - 2^l)}{3^l - 1}|E^l|$ ,  $\Sigma_{G'} = \frac{3^l - 2^l}{3^l - 1}|E^l|$ , and  $r_{G'} = |W_{G'}^-|/W_{G'}^+ = 1 - \frac{1}{2^l} = 1 - O(\delta^c)$ , where  $c$  is a positive constant.

**Completeness.** Suppose  $G^l$  is  $l$ -fold 3-colorable. Let  $\chi^l$  be the optimal  $l$ -fold 3-coloring, then  $m(\chi^l) = |E^l|$ . Let  $f = \chi^l$ . For any edge  $e = (u, v)$  in  $E^l$ ,  $f_{u,j} \neq f_{v,j}$  for any  $1 \leq j \leq l$ , since  $\chi^l$  is a  $l$ -fold 3-coloring. So the total weight of the unsatisfied edges between  $u$  and  $v$  is 0. Therefore, the total weight of the unsatisfied edges in  $E'$  is 0, i.e.  $Val(G') = 0 < \zeta$ .

**Soundness.** Suppose the value of  $G^l$  is at most  $\delta$ . For any labeling  $f$  of  $G'$ ,  $\chi^l = f$  is a  $l$ -fold 3-coloring of  $G^l$ , and  $m(\chi^l) < \delta$ . So at least  $1 - \delta$  fraction of edges in  $E^l$  are not properly  $l$ -fold 3-colored. The two vertices of these edges share the same color in at least one element, in another word, the labels of the two vertices under  $f$  share the same value in at least one element. Since the total weight of the unsatisfied edges between such two vertices in  $E'$  is 1, the total weight of the unsatisfied edges in  $E'$  under  $f$  is at least  $(1 - \delta)|E^l|$ . Therefore,  $Val(G') \geq (1 - \delta)|E^l|/\Sigma_{G'} > 1 - \delta$ .  $\square$

In the end of this section, we establish a connection from Conjecture 4 to the Unique Game Conjecture by the following theorem.

**Theorem 4.** *If Conjecture 4 holds true for any  $\rho > 0$ , Conjecture 2 holds true.*

*Proof.* Suppose Conjecture 2 holds false, then for some  $\zeta, \delta > 0$ , for any label size  $k$ , we can decide in polynomial time whether Min UGP with  $k$  labels has a value more than  $1 - \delta$  or less than  $\zeta$ . We claim Conjecture 4 for  $\rho = \min(\zeta, \delta)/2$  holds false.

Given an instance  $G = (V, E)$  of Min GUGP-PWT( $\rho$ ), we construct an instance  $G' = (V, E')$  of Min UGP as follows. Let  $E'$  be the set of the edges in  $E$  with positive weights. Let  $f^*$  be the optimal labeling of  $G$ , and  $f'$  be the optimal labeling of  $G'$ . Then  $Val(G') = W_{G'}(f')/W_G^+$  and  $Val(G) = W_G(f^*)/\Sigma_G$ .

Since  $W_{G'}(f') \geq W_G(f')$  and  $\Sigma_G/W_G^+ \geq 1 - \rho$ ,  $Val(G') \geq (1 - \rho)Val(G)$ .

By the definition of  $E'$ ,  $W_G(f^*) \geq W_{G'}(f^*) - \rho W_G^+$ . We have  $Val(G)\Sigma_G = W_G(f^*) \geq W_{G'}(f^*) - \rho W_G^+ \geq W_{G'}(f') - \rho W_G^+ = (Val(G') - \rho)W_G^+$ . Therefore,  $Val(G') \leq Val(G) + \rho$ .

If  $Val(G) < \zeta/2$ , then  $Val(G') < \zeta$ . If  $Val(G) > 1 - \delta/2$ , then  $Val(G') > 1 - \delta$ . Thus we can decide in polynomial time whether the instance of Min GUGP-PWT( $\rho$ ) has a value more than  $1 - \delta/2$  or less than  $\zeta/2$ .  $\square$

## 5 Discussions

We notice the proof of Theorem 3 does not apply to the case for any  $\rho < 1$ , since  $r_{G'} \rightarrow 1$  when  $\delta \rightarrow 0$ . We leave it an open problem whether Unique Game Conjecture holds true on GUGP-PWT( $\rho$ ) for  $0 < \rho < 1$ .

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# On Unique Games with Negative Weights

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**Abstract.** In this paper, the authors define Generalized Unique Game Problem (GUGP), where weights of the edges are allowed to be negative. Focuses are made on two special types of GUGP, GUGP-NWA, where the weights of all edges are negative, and GUGP-PWT( $\rho$ ), where the total weight of all edges are positive and the negative/positive ratio is at most  $\rho$ . The authors investigate the counterpart of the Unique Game Conjecture on GUGP-PWT( $\rho$ ). The authors prove Unique Game Conjecture holds true on GUGP-PWT(1) by reducing the parallel repetition of Max 3-Cut Problem to GUGP-PWT(1), and Unique Game Conjecture holds true on GUGP-PWT(1/2) if the 2-to-1 Conjecture holds true. The authors pose an open problem whether Unique Game Conjecture holds true on GUGP-PWT( $\rho$ ) with  $0 < \rho < 1$ .

## 1 Introduction

The Unique Game Conjecture is put forward by Khot on STOC 2002 as a powerful tool to prove lower bound of inapproximability for combinatorial optimization problems[1]. It has been shown by researchers a positive resolution of this conjecture would imply improved even best possible hardness results for many famous problems, to name a few, Max Cut, Vertex Cover, Multicut, Min 2CNF Deletion, making an important challenge to prove or refute the conjecture.

Some variations of the Unique Game Conjecture have been mentioned. Rao proves a strong parallel repetition theorem which shows Weak Unique Game Conjecture is equivalent to the Unique Game Conjecture[2]. Khot et al. show that Unique Game Conjecture on Max 2LIN( $q$ ) is equivalent to the Unique Game Conjecture[3]. Khot pose the  $d$ -to-1 Conjectures in his original paper for  $d \geq 2$ [1]. O’Donnell et al. show a tight

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hardness for approximating satisfiable constraint satisfaction problem on 3 Boolean variables assuming the  $d$ -to-1 Conjecture for any fixed  $d$ [4]. Dinur et al. use the 2-to-2 Conjecture and the  $\ltimes$  conjecture to derive the hardness results of Approximate Coloring Problem, and prove that the (exact) 2-to-1 Conjecture implies their 2-to-2 Conjecture[5]. Guruswami et al. use the (exact) 2-to-1 Conjecture to derive the hardness result of Maximum  $k$ -Colorable Subgraph Problem[6]. It is unknown whether the Unique Game Conjecture implies any of the  $d$ -to-1 Conjectures, or vice versa.

Almost concurrently to [1], Khot defines the smoothness property of 2-Prover 1-Round Game (2P1R), a weaker analogue of the property of Unique Game Problem[7]. He constructs instances of 2P1R that are  $\eta$ -smooth for arbitrarily small  $\eta$  to prove hardness of Coloring 3-Uniform Hypergraphs. 2P1R with certain smoothness properties have been used to prove some other hardness results[8,9,10].

In this paper, the authors relax Unique Game Problem (UGP) by defining Generalized Unique Game Problem (GUGP), where weights of the edges are allowed to be negative. Focuses are made on two special types of GUGP, GUGP-NWA, where the weights of all edges are negative, and GUGP-PWT( $\rho$ ), where the total weight of all edges are positive and the negative/positive ratio is at most  $\rho$ . Max GUGP-NWA can be restated as 2P1R where each relation is a complement of a permutation. GUGP-PWT( $\rho$ ) over  $1 \geq \rho \geq 0$  makes a possible phase transition from a hard  $(1 - \zeta, \delta)$ -gap problem of 2P1R to UGP.

The authors investigate the counterpart of the Unique Game Conjecture on GUGP-PWT( $\rho$ ). The authors prove Unique Game Conjecture holds true on GUGP-PWT(1) by reducing the parallel repetition of Max 3-Cut Problem to GUGP-PWT(1), and Unique Game Conjecture holds true on GUGP-PWT(1/2) if the 2-to-1 Conjecture holds true. It is shown the  $(1 - \zeta, \delta)$ -gap hardness of problems in GUGP-PWT( $\rho$ ) possesses the compactness property when  $\rho \rightarrow 0$ . The authors pose an open problem whether Unique Game Conjecture holds true on GUGP-PWT( $\rho$ ) with  $0 < \rho < 1$ .

Section 2 demonstrates some definitions and conjectures. The authors show the main results for GUGP-NWA in Section 3 and for GUGP-PWT( $\rho$ ) in Section 4. Section 5 is some discussions.

## 2 Preliminaries

In 2-Prover 1-Round Game Problem (2P1R), we are given a bipartite graph  $G = (V, W; E)$ , with each edge  $e$  having a weight  $w_e \in \mathbb{Q}^+$ . We are also given two sets of labels,  $k_1$  and  $k_2$ , which we identify with  $[k_1] = \{1, \dots, k_1\}$  and  $[k_2] = \{1, \dots, k_2\}$ . Each edge  $e = (u, v)$  in the graph is equipped with a relation  $R_e \subseteq [k_1] \times [k_2]$ . The solution of the problem is a labeling  $f_1 : V \rightarrow [k_1]$  and  $f_2 : W \rightarrow [k_2]$  which assigns a label to each vertex of  $G$ . An edge  $e = (u, v)$  is said to be satisfied under  $f_1$  and  $f_2$  if  $(f_1(u), f_2(v)) \in R_e$ , else is said to be unsatisfied. The object of the problem is to find a labeling maximizing the total weight of the satisfied edges. The value of the instance,  $Val(G)$ , is defined as the maximum total weight of the satisfied edges divided by the total weight of all edges.

An instance of 2P1R has the projection property if all relations which the edges are equipped with are projections. We say an instance of 2P1R with the projection property is  $\eta$ -smooth (or has  $\eta$ -smoothness property) if for every  $u \in V$  and two distinct labels

$i, j \in [k_1]$ , we have  $Pr_v[\sigma_{(u,v)}(i) = \sigma_{(u,v)}(j)] \leq \eta$ , where  $v$  is a randomly chosen neighbor of  $u$ , and  $\sigma_{(u,v)}$  is the projection which the edge  $(u, v)$  is equipped with.

The following proposition describes the hardness of 2P1R with the smoothness property:

**Proposition 1. ([10] Theorem 3.5)** *For every  $\eta, \delta > 0$ , there are  $k_1 = k_1(\eta, \delta)$  and  $k_2 = k_2(\eta, \delta)$  such that given an  $\eta$ -smooth instance  $G$  of 2P1R with the label sets  $[k_1]$  and  $[k_2]$  it is NP-hard to distinguish whether  $Val(G) = 1$  or  $Val(G) < \delta$ .*

Unique Game Problem (UGP) can be viewed as 2P1R with 0-smoothness property. In UGP, we are given an graph  $G = (V, E)$ , a weight function  $w_e \in \mathbb{Q}^+$  for  $e \in E$ , and a set of labels,  $[k]$ . Each edge  $e = (u, v)$  in the graph is equipped with a permutation  $\pi_e : [k] \rightarrow [k]$ . The solution of the problem is a labeling  $f : V \rightarrow [k]$  which assigns a label to each vertex of  $G$ . An edge  $e = (u, v)$  is said to be satisfied under  $f$  if  $\pi_e(f(u)) = f(v)$ , else is said to be unsatisfied. Note that we allow  $G$  is a graph with parallel edges, i.e., there exist more than one edge between two vertices.

It is possible to define two optimization problems in this situation. In Max UGP, the value of the instance is defined as the maximum total weight of the satisfied edges divided by the total weight of all edges. In Min UGP, the value of the instance is defined as the minimum total weight of the unsatisfied edges divided by the total weight of all edges.

Khot initiates much of the interest in the following conjecture by showing that many hardness results stem from it. It basically states that it is NP-hard to distinguish whether many or only few edges are satisfied.

**Conjecture 1. ([1] Unique Game Conjecture in Max UGP Form)** *For every  $\zeta, \delta > 0$ , there is a  $k = k(\zeta, \delta)$  such that given an instance  $G$  of Max UGP with  $k$  labels it is NP-hard to distinguish whether  $Val(G) > 1 - \zeta$  or  $Val(G) < \delta$ .*

The conjecture can be restated in Min UGP form, and the two conjectures are equivalent.

**Conjecture 2. (Unique Game Conjecture in Min UGP Form)** *For every  $\zeta, \delta > 0$ , there is a  $k = k(\zeta, \delta)$  such that given an instance  $G$  of Min UGP with  $k$  labels it is NP-hard to distinguish whether  $Val(G) < \zeta$  or  $Val(G) > 1 - \delta$ .*

2-to-1 Game and 2-to-2 Game are two special types of 2P1R. In 2-to-1 Game, we are given a bipartite graph  $G = (V, W; E)$ , with each edge  $e$  having a weight  $w_e \in \mathbb{Q}^+$ . We are also given two sets of labels,  $[2k]$  for  $V$  and  $[k]$  for  $W$ . Each edge  $e = (u, v)$  in the graph is equipped with a 2-to-1 projection. A projection  $\sigma : [2k] \rightarrow [k]$  is said to be a 2-to-1 projection if for each element  $j \in [k]$  we have  $|\sigma^{-1}(j)| = 2$ . The value of the instance of 2-to-1 Game,  $Val(G)$ , is defined as the minimum total weight of the unsatisfied edges divided by the total weight of all edges.

In 2-to-2 Game, we are given an graph  $G = (V, E)$ , a weight function  $w_e \in \mathbb{Q}^+$  for  $e \in E$ , and a set of labels,  $[k]$ . Each edge  $e = (u, v)$  in the graph is equipped with a

2-to-2 relation. A relation  $R \subseteq [2k] \times [2k]$  is said to be a 2-to-2 relation if there are two permutations  $\pi_u, \pi_v : [2k] \rightarrow [2k]$  such that  $(i, j) \in R$  iff  $(\pi_u(i), \pi_v(j)) \in T$  where

$$T := \bigcup_{l=1}^k \{(2l-1, 2l-1), (2l-1, 2l), (2l, 2l-1), (2l, 2l)\}.$$

The value of the instance of 2-to-2 Game,  $Val(G)$ , is defined as the minimum total weight of the unsatisfied edges divided by the total weight of all edges.

The authors list the (exact) 2-to-1 Conjecture and the 2-to-2 Conjecture in their minimization forms. The latter is somewhat different from that in [5]. It can be proved that the 2-to-1 Conjecture implies our 2-to-2 Conjecture along the line of [5].

**Conjecture 3. (2-to-1 Conjecture)** *For every  $\delta > 0$ , there is a  $k = k(\delta)$  such that given an instance  $G$  of 2-to-1 Game with the label sets  $[2k]$  and  $[k]$  it is NP-hard to distinguish whether  $Val(G) = 0$  or  $Val(G) > 1 - \delta$ .*

**Conjecture 4. (2-to-2 Conjecture)** *For every  $\delta > 0$ , there is a  $k = k(\delta)$  such that given an instance of 2-to-2 Game with the label set  $[2k]$  it is NP-hard to distinguish whether  $Val(G) = 0$  or  $Val(G) > 1 - \delta$ .*

In this paper, the authors define Generalized Unique Game Problem (GUGP), where weights of the edges are allowed to be negative. In GUGP, we are given an graph  $G = (V, E)$  possibly having parallel edges, weight function  $w_e \in \mathbb{Q}$  for  $e \in E$ , and the set of labels,  $[k]$ . Each edge  $e = (u, v)$  in the graph is equipped with a permutation  $\pi_e : [k] \rightarrow [k]$ .

Note that  $w_e$  could be positive or negative. We assume there is no edge with zero weight for sake of clearance. Let  $W_G^+$  be the total of the positive weights of all edges,  $W_G^-$  be the total of the negative weights of all edges, and  $\Sigma_G = W_G^+ + W_G^-$  be the total weight of all edges. We call  $r_G = |W_G^-|/W_G^+$  the negative/positive ratio of the instance.

The solution of GUGP is a labeling  $f : V \rightarrow [k]$  which assigns a label to each vertex of  $G$ . It is also possible to define two optimization problems. In Max GUGP, the goal of is to maximize the total weight of the satisfied edges. In Min GUGP, the goal is to minimize the total weight of the unsatisfied edges. GUGP-NWA and GUGP-PWT are two special types of GUGP. In GUGP-NWA, the weight of all edges are negative. In GUGP-PWT, the total weight of all edges is positive.

In Max GUGP-NWA, we seek to minimize the total weight of the unsatisfied edges, i.e. to maximize the *absolute value* of the total weight of the unsatisfied edges. The value of Max GUGP-NWA is defined as the maximum absolute value of the total weight of the unsatisfied edges divided by  $|W_G^-|$ .

In Min GUGP-NWA, we seek to maximize the total weight of the satisfied edges, i.e. to minimize the *absolute value* of the total weight of the satisfied edges. The value of Min GUGP-NWA is defined as the minimum absolute value of the total weight of the satisfied edges divided by  $|W_G^-|$ .

In Max GUGP-PWT, we seek to maximize the total weight of the satisfied edges. The value of Max GUGP-PWT is defined as the maximum total weight of the satisfied edges divided by  $\Sigma_G$ . In an instance  $G$  of Max GUGP-PWT, let  $W_G(f)$  be the total

weight of the satisfied edges under labeling  $f$ , let the optimal labeling be  $f^*$ . The value of the instance is  $Val(G) = W_G(f^*)/\Sigma_G$ .

In Min GUGP-PWT, we seek to minimize the total weight of the unsatisfied edges. The value of Min GUGP-PWT is defined as the minimum total weight of the unsatisfied edges divided by  $\Sigma_G$ . In an instance  $G$  of Min GUGP-PWT, let  $W_G(f)$  be the total weight of the unsatisfied edges under labeling  $f$ , let the optimal labeling be  $f^*$ . The value of the instance is  $Val(G) = W_G(f^*)/\Sigma_G$ .

We remind the reader that the value of Max GUGP-PWT and Min GUGP-PWT could be negative or more than 1, although we still use the customary words "Completeness" and "Soundness" in our proofs of Theorem 3 and Theorem 4.

We define Max/Min GUGP-PWT( $\rho$ ) as the subproblem of Max/Min GUGP-PWT where the negative/positive ratio of the instances is upper bounded by  $\rho$ , where  $\rho$  is a constant independent from  $k$ . Since the negative/positive ratio is always less than 1, we set the range of  $\rho$  to be  $0 \leq \rho \leq 1$ . Note that Max/Min GUGP-PWT(0) is just Max/Min UGP.

We give the two equivalent counterparts of the Unique Game Conjecture on Max GUGP-PWT( $\rho$ ) and Min GUGP-PWT( $\rho$ ) as follows:

**Conjecture 5. (Unique Game Conjecture on Max GUGP-PWT( $\rho$ ))** *For every  $\zeta, \delta > 0$ , there is a  $k = k(\zeta, \delta)$  such that given an instance of Max GUGP-PWT( $\rho$ ) with  $k$  labels it is NP-hard to distinguish whether  $Val(G) > 1 - \zeta$  or  $Val(G) < \delta$ .*

**Conjecture 6. (Unique Game Conjecture on Min GUGP-PWT( $\rho$ ))** *For every  $\zeta, \delta > 0$ , there is a  $k = k(\zeta, \delta)$  such that given an instance of Min GUGP-PWT( $\rho$ ) with  $k$  labels it is NP-hard to distinguish whether  $Val(G) < \zeta$  or  $Val(G) > 1 - \delta$ .*

The conjectures states it is NP-hard to distinguish the following two cases: there is a labeling under which the absolute value of the total of the negative weight of the unsatisfied edges is almost no less than the total of the positive weight of the unsatisfied edges; under any labeling the absolute value of the total of the negative weight of the satisfied edges is almost no less than the total of the positive weight of the satisfied edges.

### 3 GUGP-NWA

In this section, we prove it is NP-hard to approximate Min GUGP-NWA within any factor of  $poly(n)$ , and we prove Max GUGP-NWA can be approximated with factor 2.

**Theorem 1.** *It is NP-hard to approximate Min GUGP-NWA within any factor of  $poly(n)$ .*

*Proof.* Min GUGP-NWA can be restated as: In the situation of UGP, the goal is to find minimum fraction of the total weight of the *satisfied edges*. We construct an approximation ratio preservation reduction from TSP to the above problem.

Given an instance of TSP problem  $G = (V, E)$ , where each edge of  $E$  has a weight  $w_e \in \mathbb{Q}^+$ . Denote  $n := |V|$ . The instance of the restated form of Min GUGP-NWA is a graph  $G' = G'(V, E')$ , with each edge  $e' \in E'$  having a weight  $w'(e')$ , and with

the labeling set  $[n]$ . For each edge  $e = (u, v) \in E$ , there are three parallel edges  $e^-$ ,  $e^+$  and  $e^-$  between  $u$  and  $v$  in  $E'$ .  $e^-$  has weight  $M$  and equipped with permutation  $\pi^- = \{(1, 1), (2, 2), \dots, (n, n)\}$ . Let  $M = n \cdot \text{Max}(w)$ , where  $\text{Max}(w)$  is the maximum weight of all edges in  $G$ .  $e^+$  has weight  $w(e)$  and equipped with permutation  $\pi^+ = \{(1, 2), (2, 3), \dots, (n, 1)\}$ .  $e^-$  has weight  $w(e)$  and equipped with permutation  $\pi^- = \{(1, n), (2, 1), \dots, (n, n-1)\}$ .

Given a solution of TSP problem, a Hamiltonian cycle  $C$ , we can assign label 1 to  $n$  to vertices of  $C$  along  $C$  in  $G'$ , and the total weight of satisfied edges in  $G'$  is exactly the total weight of edges on  $C$  in  $G$ .

In the other direction, given a labeling  $f$  of  $G'$ , if there are two vertices assigned with the same label, the total weight of the satisfied edges is at least  $M$ . Otherwise all vertices are assigned with label from 1 to  $n$  respectively, let  $u_i$  be the vertices assigned label  $i$  for  $1 \leq i \leq n$ , and  $e_i^+ \in E'$  be the edge between  $u_i$  and  $u_{i+1 \bmod n}$  equipped with permutation  $\pi^+$ . The total weight of the satisfied edges is equal to  $\sum_{1 \leq i \leq n} w'(e_i^+)$ . Let  $C$  be the Hamiltonian cycle of  $G$  which consists of vertices from  $u_1$  to  $u_n$ , then the total weight of  $C$  in  $G$  is exactly the total weight of satisfied edges under  $f$  in  $G'$ .  $\square$

**Theorem 2.** *Max GUGP-NWA can be approximated within factor 2.*

*Proof.* Max GUGP-NWA can be restated as the following 2P1R. We are given an graph  $G = (V, E)$ , a weight function  $w_e \in \mathbb{Q}^+$ , and the set of labels,  $[k]$ . Each edge  $e = (u, v)$  in the graph is equipped with a relation  $\bar{\pi}_e = [k] \times [k] - \pi_e$ , where  $\pi_e : [k] \rightarrow [k]$  is a permutation. The solution of the problem is a labeling  $f : V \rightarrow [k]$  which assigns a label to each vertex of  $G$ . An edge  $e = (u, v)$  is said to be satisfied under  $f$  if  $(f(u), f(v)) \in \bar{\pi}_e$ . The value of the instance is defined as the maximum fraction of the total weight of the satisfied edges

We describe an approximation algorithm that finds a solution under which the fraction of the total weight of the satisfied edges is at least  $1/2$ , which is at least half of the value of the instance.

In the beginning of the algorithm, assign arbitrary labels to all vertices. Let  $\text{Ass}(v, e)$  be the predicate whether  $v \in V$  is associated with  $e \in E$ , and  $\text{Sat}(e)$  be the predicate whether  $e$  is satisfied by current labeling. In each iteration of the algorithm, let  $U^< = \{v \in V \mid \sum_{\text{Ass}(v, e) \wedge \text{Sat}(e)} w_e < \frac{1}{2} \sum_{\text{Ass}(v, e)} w_e\}$ , and  $U^\geq = \{v \in V \mid \sum_{\text{Ass}(v, e) \wedge \text{Sat}(e)} w_e \geq \frac{1}{2} \sum_{\text{Ass}(v, e)} w_e\}$ . If  $U^< = \emptyset$ , the algorithm stops. Otherwise, choose a vertex  $u$  from  $U^<$ , suppose the label assigned to  $u$  is  $f_1$ , assign another label  $f_2$  to the vertex. If an edge  $e = (u, v)$  is unsatisfied under the old labeling, it must be the case  $(f_1, f(v)) \notin \bar{\pi}_e$ , and  $\pi_e(f_1) = f(v)$ . So  $\pi_e(f_2) \neq f(v)$ , and  $(f_2, f(v)) \in \bar{\pi}_e$ . Therefore, in the new labeling vertex  $u$  satisfies the condition of  $U^\geq$ , and we move it from  $U^<$  to  $U^\geq$ . Since after each iteration, the number of vertices in  $U^\geq$  is increased by 1, the algorithm stops in  $|V|$  iterations.  $\square$

## 4 GUGP-PWT( $\rho$ )

### 4.1 Parallel Repetition of Max 3-Cut

In Max 3-Cut Problem, the instance is a graph  $G = (V, E)$ , the value of the instance is the maximum fraction of properly colored edges of  $G$  under a 3-coloring. 3-coloring

of a graph in a color set  $[3]$  is a function  $\chi : V \rightarrow [3]$ . We say an edge is properly colored under a 3-coloring if its endpoints receive distinct colors. A graph is 3-colorable if there is a coloring under which all edges are properly colored. Max 3-Cut Problem can be viewed as a 2P1R with  $k = 3$ , where each edge is equipped with a relation  $\pi_e = \{(1, 2), (2, 3), (3, 1), (2, 1), (3, 2), (1, 3)\}$ .

Given an instance of Max 3-Cut Problem,  $G$ , we define the  $l$ -fold parallel repetition of the instance,  $G^l = G^l(V^l, E^l)$ , as follows.  $G^l = \{ \langle u_1, \dots, u_l \rangle \mid u_i \in V, 1 \leq i \leq l \}$ , and  $E^l = \{ \langle \langle u_1, \dots, u_l \rangle, \langle v_1, \dots, v_l \rangle \rangle \mid (u_i, v_i) \in E, 1 \leq i \leq l \}$ . The value of the instance is the maximum fraction of properly colored edges under a  $l$ -fold 3-coloring. Define a  $l$ -fold 3-coloring of  $G^l$  in the color set  $[3]^l$  be the function  $\chi^l : V \rightarrow [3]^l$ , and let  $m(\chi^l)$  be the number of properly colored edges under  $\chi^l$ . We say an edge  $e = \langle \langle u_1, \dots, u_l \rangle, \langle v_1, \dots, v_l \rangle \rangle$  in  $E^l$  is properly colored under a  $l$ -fold 3-coloring if  $\chi(u_i) \neq \chi(v_i)$  for any  $1 \leq i \leq l$ . The graph  $G^l$  is 3-colorable if there is a  $l$ -fold 3-coloring under which all edges are properly colored.

Petrant [11] shows that Max 3-Cut Problem possesses a hard gap at location 1, i.e. it is NP-hard to distinguish whether the instance is 3-colorable or whether has value at most  $1 - \gamma$  for some constant  $\gamma$ . The constant is presumably very small and not determined in his paper. V. Guruswami et al. [6] make the constant clear to be  $1/33 - \alpha$  for any  $\alpha > 0$ .

**Lemma 1.** ([11] Theorem 3.3) *It is NP-hard to distinguish whether the instance of Max 3-Cut Problem is whether 3-colorable or has value at most  $1 - \gamma$  for some constant  $\gamma > 0$ .*

Raz's Parallel Repetition Theorem is used to enlarge the gap of 2P1R with perfect completeness. We introduce Lemma 2 when applying Parallel Repetition Theorem to  $l$ -fold parallel repetition of Max 3-Cut Problem, and get Lemma 3 by a gap-reduction.

**Lemma 2.** ([12] Theorem 1.1) *If an instance of Max 3-Cut Problem has value  $1 - \gamma$ , the value of the  $l$ -fold parallel repetition of the instance is at most  $(1 - \gamma^{c_1})^{c_2 l}$ , where  $c_1$  and  $c_2$  are two positive constants.*

**Lemma 3.** *For any constant  $\delta > 0$ , there is a constant  $l = l(\delta)$  such that it is NP-hard to distinguish whether the instance of  $l$ -fold parallel repetition of Max 3-Cut Problem is  $l$ -fold 3-colorable or has value at most  $\delta$ .*

*Proof.* Given an instance of Max 3-Cut Problem,  $G = (V, E)$ , let  $l$  be the integer no less than  $\frac{\ln \delta}{c_2 \ln(1 - \gamma^{c_1})}$ . Let  $G^l$  be the  $l$ -fold parallel repetition of  $G$ . The proof can be achieved by the following two steps and Lemma 2.

**Completeness.** Suppose  $G$  is 3-colorable. Define a  $l$ -fold 3-coloring of  $G^l$  as  $\chi^l(\langle v_1, \dots, v_l \rangle) = \langle \chi^*(v_1), \dots, \chi^*(v_l) \rangle$ , where  $\chi^*$  is the optimal 3-coloring of  $G$ . Since all edges in  $G$  are properly colored under  $\chi^*$ , all edges in  $G^l$  are properly colored under  $\chi^l$ . Therefore,  $G^l$  is  $l$ -fold 3-colorable.

**Soundness.** Suppose  $G$  has value at most  $1 - \gamma$ . By Lemma 2, the value of  $G^l$  is at most  $(1 - \gamma^{c_1})^{c_2 l}$ , which is at most  $\delta$  by the definition of  $l$ .  $\square$

## 4.2 Unique Game Conjecture on GUGP-PWT( $\rho$ )

Let us show Conjecture 6 holds true at the boundary of the range of  $\rho$ .

**Theorem 3.** *Conjecture 6 holds true for  $\rho = 1$ .*

*Proof.* Given  $\zeta, \delta > 0$ , let  $G^l = (V^l, E^l)$  be an instance of the parallel repetition of the Max 3-Cut Problem. The instance of Min GUGP-PWT(1) is a graph  $G' = (V^l, E')$ , with labeling set  $[3^l]$ .

To accomplish the reduction, we design a gadget as replacing each edge  $e = (u, v)$  in  $E^l$  with  $3^l$  parallel edges  $e_{i_1}, \dots, e_{i_l}$  for  $1 \leq i_j \leq 3, 1 \leq j \leq l$  between  $u$  and  $v$  in  $E'$ . Let  $E^=$  be the set of edges such that at least one index is 1, and  $E^\neq$  be the set of edges such that all indexes are 2 or 3. Note that  $|E^=| = 3^l - 2^l$  and  $|E^\neq| = 2^l$ . Edges in  $E^=$  has weight  $w_x$ , and edges in  $E^\neq$  has weight  $w_y$ .  $e_{i_1, \dots, i_l}$  for  $1 \leq i_j \leq 3, 1 \leq j \leq l$  is equipped with permutation  $\pi_{i_1, \dots, i_l} = \{(\langle f_1, \dots, f_l \rangle, \langle f_1 + i_1 - 1 \bmod 3, \dots, f_l + i_l - 1 \bmod 3 \rangle) | f_j \in [3], 1 \leq j \leq l\}$ .

Note that there is always exactly one satisfied edge in  $E'$  between  $u$  and  $v$  under any labeling of  $G'$ . Suppose two vertices  $u$  and  $v$  are assigned labels  $\langle f_{u,1}, \dots, f_{u,l} \rangle$  and  $\langle f_{v,1}, \dots, f_{v,l} \rangle$  respectively. We require: (i) when  $f_{u,j} = f_{v,j}$  for at least one  $1 \leq j \leq l$ , the total weight of the unsatisfied edges between  $u$  and  $v$  in  $E'$  is 1; (ii) when  $f_{u,j} \neq f_{v,j}$  for any  $1 \leq j \leq l$ , the total weight of the unsatisfied edges between  $u$  and  $v$  in  $E'$  is 0.

Let us determine the value of  $w_x$  and  $w_y$ . By the linear equations,

$$\begin{cases} (3^l - 2^l - 1)w_x + 2^l w_y = 1 \\ (3^l - 2^l)w_x + (2^l - 1)w_y = 0 \end{cases}$$

we have

$$\begin{cases} w_x = -\frac{2^l - 1}{3^l - 1} \\ w_y = \frac{3^l - 2^l}{3^l - 1} \end{cases}.$$

Note that  $w_x < 0$  and  $w_y > 0$ .  $W_{G'}^+ = \frac{2^l(3^l - 2^l)}{3^l - 1}|E^l|$ ,  $W_{G'}^- = -\frac{(2^l - 1)(3^l - 2^l)}{3^l - 1}|E^l|$ ,  $\Sigma_{G'} = \frac{3^l - 2^l}{3^l - 1}|E^l|$ , and  $r_{G'} = |W_{G'}^-|/W_{G'}^+ = 1 - \frac{1}{2^l} = 1 - O(\delta^c)$ , where  $c$  is a positive constant.  $r_{G'} \rightarrow 1$  when  $\delta \rightarrow 0$ .

The proof is completed by the following two steps and Lemma 3.

**Completeness.** Suppose  $G^l$  is  $l$ -fold 3-colorable. Let  $\chi^l$  be the optimal  $l$ -fold 3-coloring, then  $m(\chi^l) = |E^l|$ . Let  $f = \chi^l$ . For any edge  $e = (u, v)$  in  $E^l$ ,  $f_{u,j} \neq f_{v,j}$  for any  $1 \leq j \leq l$ , since  $\chi^l$  is a  $l$ -fold 3-coloring. So the total weight of the unsatisfied edges between  $u$  and  $v$  is 0. Therefore, the total weight of the unsatisfied edges in  $E'$  is 0, i.e.  $Val(G') = 0 < \zeta$ .

**Soundness.** Suppose the value of  $G^l$  is at most  $\delta$ . For any labeling  $f$  of  $G'$ ,  $\chi^l = f$  is a  $l$ -fold 3-coloring of  $G^l$ , and  $m(\chi^l) < \delta$ . So at least  $1 - \delta$  fraction of edges in  $E^l$  are not properly  $l$ -fold 3-colored. The two vertices of these edges share the same color in at least one element, in another word, the labels of the two vertices under  $f$  share the same



value in at least one element. Since the total weight of the unsatisfied edges between such two vertices in  $E'$  is 1, the total weight of the unsatisfied edges in  $E'$  under  $f$  is at least  $(1 - \delta)|E^l|$ . Therefore,  $Val(G') \geq (1 - \delta)|E^l|/\Sigma_{G'} > 1 - \delta$ .  $\square$

We prove that Conjecture 6 holds true for  $\rho = 1/2$  if Conjecture 4 holds true. Since Conjecture 3 implies Conjecture 4, Conjecture 6 holds true for  $\rho = 1/2$  if Conjecture 3 holds true.

For any two positive integers  $m$  and  $n$ , let  $r$  be the reminder when dividing  $m$  by  $n$ , then  $0 \leq r \leq n - 1$ . Let  $m \bmod^+ n = r$ , if  $r > 0$ ;  $n$ , if  $r = 0$ .

**Theorem 4.** *Conjecture 6 holds true for  $\rho = 1/2$  if Conjecture 4 holds true.*

*Proof.* Given  $\zeta, \delta > 0$ , let  $G = (V, E)$  be an instance of 2-to-2 Game Problem, with labeling set  $[2k]$ . We construct an instance of Min GUGP-PWT(1/2) as a graph  $G' = (V, E')$ , with labeling set  $[2k]$ . For each edge  $e = (u, v)$  in  $E$  with the 2-to-2 relation  $R$ , let the two permutations w.r.t.  $R$  are  $\pi_u, \pi_v$ , we design a gadget as replacing  $e$  with  $2k$  parallel edges  $e_1, \dots, e_{2k}$  between  $u$  and  $v$  in  $E'$ .

The edge  $e_1$  has weight  $w_x$  and is equipped with the permutation

$$\pi_1 = \{(\pi_u^{-1}(1), \pi_v^{-1}(1)), (\pi_u^{-1}(2), \pi_v^{-1}(2)), \dots, (\pi_u^{-1}(2k-1), \pi_v^{-1}(2k-1)), (\pi_u^{-1}(2k), \pi_v^{-1}(2k))\}.$$

The edge  $e_2$  has weight  $w_x$  and is equipped with the permutation

$$\pi_2 = \{(\pi_u^{-1}(1), \pi_v^{-1}(2)), (\pi_u^{-1}(2), \pi_v^{-1}(1)), \dots, (\pi_u^{-1}(2k-1), \pi_v^{-1}(2k)), (\pi_u^{-1}(2k), \pi_v^{-1}(2k-1))\}.$$

The edge  $e_{2j-1}$  for  $2 \leq j \leq k$  has weight  $w_y$  and is equipped with the permutation

$$\pi_{2j-1} = \bigcup_{i=1}^k \{(\pi_u^{-1}(2i-1), \pi_v^{-1}(2i-1+2j-2 \bmod^+ 2k)), (\pi_u^{-1}(2i), \pi_v^{-1}(2i+2j-2 \bmod^+ 2k))\}.$$

The edge  $e_{2j}$  for  $2 \leq j \leq k$  has weight  $w_y$  and is equipped with the permutation

$$\pi_{2j} = \bigcup_{i=1}^k \{(\pi_u^{-1}(2i-1), \pi_v^{-1}(2i-1+2j-1 \bmod^+ 2k)), (\pi_u^{-1}(2i), \pi_v^{-1}(2i+2j-3 \bmod^+ 2k))\}.$$

Note that there is always exactly one satisfied edge in  $E'$  between  $u$  and  $v$  under any labeling of  $G'$ . Suppose two vertices  $u$  and  $v$  are assigned labels  $f_u$  and  $f_v$  respectively. We require: (i) when the edge  $e = (u, v)$  is satisfied under  $f$  in  $G$ , i.e. one of the two edges  $e_1$  and  $e_2$  is satisfied, the total weight of the unsatisfied edges between  $u$  and  $v$  in  $E'$  is 1; (ii) when the edge  $e = (u, v)$  is unsatisfied under  $f$  in  $G$ , i.e. one of the edges  $e_{2j-1}$  and  $e_{2j}$  for  $2 \leq j \leq k$  is satisfied, the total weight of the unsatisfied edges between  $u$  and  $v$  in  $E'$  is 0.

Let us determine the value of  $w_x$  and  $w_y$ . By the linear equations

$$\begin{cases} w_x + (2k-2)w_y = 0 \\ 2w_x + (2k-3)w_y = 1 \end{cases}$$

we have

$$\begin{cases} w_x = \frac{2k-2}{2k-1} \\ w_y = -\frac{1}{2k-1} \end{cases}.$$

Note that  $w_x > 0$ ,  $w_y < 0$ .  $W_{G'}^+ = \frac{4(k-1)}{2k-1}$ ,  $W_{G'}^- = \frac{-2(k-1)}{2k-1}$ ,  $\Sigma_{G'} = \frac{2k-2}{2k-1}|E|$  and  $r_{G'} = 1/2$ . The proof is completed by the following two steps.

**Completeness.** Suppose  $Val(G) = 0$ . Let  $f$  be the optimal labeling of  $G$ , then  $f$  is also a labeling of  $G'$ . Since any edge  $e = (u, v)$  in  $E$  is satisfied under  $f$ , the total weight of the unsatisfied edges between  $u$  and  $v$  in  $E'$  is 0. Therefore, the total weight of the unsatisfied edges in  $E'$  is 0, i.e.  $Val(G') = 0 < \zeta$ .

**Soundness.** Suppose  $Val(G) > 1 - \delta$ . Then for any labeling  $f$  of  $G'$ ,  $f$  is a labeling of  $G$ , at least  $1 - \delta$  fraction of the edges in  $E$  are unsatisfied. By the definition of  $E'$ , the total weight of the unsatisfied edges in  $E'$  between the two endpoints of such edges is 1. So the total weight of the unsatisfied edges in  $E'$  under  $f$  is at least  $(1 - \delta)|E|$ . Therefore,  $Val(G') \geq (1 - \delta)|E|/\Sigma_{G'} > 1 - \delta$ .  $\square$

In the end of this section, we establish a connection from Conjecture 6 to the Unique Game Conjecture by the following theorem.

**Theorem 5.** *If Conjecture 6 holds true for any  $\rho > 0$ , Conjecture 2 holds true.*

*Proof.* Suppose Conjecture 2 holds false, then for some  $\zeta, \delta > 0$ , for any label size  $k$ , we can decide in polynomial time whether an instance of Min UGP with  $k$  labels has a value more than  $1 - \delta$  or less than  $\zeta$ . We claim Conjecture 6 for  $\rho = \min(\zeta, \delta)/2$  holds false.

Given an instance  $G = (V, E)$  of Min GUGP-PWT( $\rho$ ), we construct an instance  $G' = (V, E')$  of Min UGP as follows. Let  $E'$  be the set of the edges in  $E$  with positive weights. Let  $f^*$  be the optimal labeling of  $G$ , and  $f'$  be the optimal labeling of  $G'$ . Then  $Val(G') = W_{G'}(f')/W_G^+$  and  $Val(G) = W_G(f^*)/\Sigma_G$ .

Since  $W_{G'}(f') \geq W_G(f') \geq W_G(f^*)$  and  $\Sigma_G/W_G^+ \geq 1 - \rho$ ,  $Val(G') \geq (1 - \rho)Val(G)$ .

By the definition of  $E'$ ,  $W_G(f^*) \geq W_{G'}(f^*) - \rho W_G^+$ . We have  $Val(G)\Sigma_G = W_G(f^*) \geq W_{G'}(f^*) - \rho W_G^+ \geq W_{G'}(f') - \rho W_G^+ = (Val(G') - \rho)W_G^+$ . Therefore,  $Val(G') \leq Val(G) + \rho$ .

If  $Val(G) < \zeta/2$ , then  $Val(G') < \zeta$ . If  $Val(G) > 1 - \delta/2$ , then  $Val(G') > 1 - \delta$ . Thus we can decide in polynomial time whether the instance of Min GUGP-PWT( $\rho$ ) has a value more than  $1 - \delta/2$  or less than  $\zeta/2$ .  $\square$

## 5 Discussions

The topic in this paper is similar to Khot's smoothness property in that both discuss on an analogue of UGP with proven  $(1 - \zeta, \delta)$ -gap hardness. It leaves as an open problem whether Unique Game Conjecture holds true on GUGP-PWT( $\rho$ ) for  $0 < \rho < 1$ . We make a reasonable and rather bold conjecture: if Conjecture 6 holds true for some  $0 < \rho < 1$ , it holds true for any  $0 < \rho < 1$ , which would lead to the corollary that the

d-to-1 Conjecture implies the Unique Game Conjecture, by Theorem 4 and Theorem 5. To confirm our conjecture, it would be very interesting to seek techniques to derive the  $(1 - \zeta, \delta)$ -gap hardness result on smaller  $\rho$  by the hardness result on larger  $\rho$ .

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# On Unique Games with Negative Weights

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**Abstract.** In this paper, the authors define Generalized Unique Game Problem (GUGP), where weights of the edges are allowed to be negative. Focuses are made on two special types of GUGP, GUGP-NWA, where the weights of all edges are negative, and GUGP-PWT( $\rho$ ), where the total weight of all edges are positive and the negative/positive ratio is at most  $\rho$ . The authors investigate the counterparts of the Unique Game Conjecture for the two types of generalized unique game problems. The authors prove Unique Game Conjecture holds false on GUGP-NWA by giving a factor 2 approximation algorithm for Max GUGP-NWA, Unique Game Conjecture holds true on GUGP-PWT(1) by reducing the parallel repetition of Max 3-Cut Problem to GUGP-PWT(1), and Unique Game Conjecture holds true on GUGP-PWT(1/2) if the 2-to-1 Conjecture holds true. The authors pose an open problem whether Unique Game Conjecture holds true on GUGP-PWT( $\rho$ ) with  $0 < \rho < 1$ .

## 1 Introduction

The Unique Game Conjecture is put forward by Khot on STOC 2002 as a powerful tool to prove lower bound of inapproximability for combinatorial optimization problems[1]. It has been shown by researchers a positive resolution of this conjecture would imply improved even best possible hardness results for many famous problems, to name a few, Max Cut, Vertex Cover, Multicut, Min 2CNF Deletion, making an important challenge to prove or refute the conjecture.

Some variations of the Unique Game Conjecture have been mentioned. Rao proves a strong parallel repetition theorem which shows Weak Unique Game Conjecture is equivalent to the Unique Game Conjecture[2]. Khot et al. show that the Unique Game Conjecture on Max 2LIN(q) is equivalent to Unique Game Conjecture[3]. Arora et al. show the Unique Game Conjecture cannot hold when the constrain graph is an expander beyond a certain threshold, and put forward the Unique Game Conjecture with

Expansion[4]. Bansal et al. use the Unique Game Conjecture with Vertex Expansion in [5].

Khot pose the  $d$ -to-1 Conjectures in his original paper[1] for  $d \geq 2$ . O'Donnell et al. show a tight hardness for approximating satisfiable constraint satisfaction problem on 3 Boolean variables assuming the  $d$ -to-1 Conjecture for any fixed  $d$ [6]. Dinur et al. use the 2-to-2 Conjecture and the  $\ltimes$  conjecture to derive the hardness results of Approximate Coloring Problem, and prove that Khot's  $d$ -to-1 Conjecture implies their  $d$ -to- $d$  Conjecture[7]. Guruswami et al. use the 2-to-1 Conjecture to derive the hardness result of Maximum  $k$ -Colorable Subgraph Problem[8]. It is unknown whether the Unique Game Conjecture implies any of the  $d$ -to-1 Conjectures, or vice versa.

In this paper, the authors relax Unique Game Problem (UGP) by defining Generalized Unique Game Problem (GUGP), where weights of the edges are allowed to be negative. Focuses are made on two special types of GUGP, GUGP-NWA, where the weights of all edges are negative, and GUGP-PWT( $\rho$ ), where the total weight of all edges are positive and the negative/positive ratio is at most  $\rho$ . Max GUGP-NWA can be restated as 2-Prover 1-Round Problem where each relation is a complement of a permutation. GUGP-PWT( $\rho$ ) over  $1 \geq \rho \geq 0$  forms a phase transition from a hard  $(1 - \zeta, \delta)$ -gap problem of 2-Prover 1-Round Problem to Min UGP.

The authors investigate the counterparts of the Unique Game Conjecture for the two types of generalized unique game problems. The authors prove Unique Game Conjecture holds false on GUGP-NWA by giving a factor 2 approximation algorithm for Max GUGP-NWA, Unique Game Conjecture holds true on GUGP-PWT(1) by reducing the parallel repetition of Max 3-Cut Problem to GUGP-PWT(1), and Unique Game Conjecture holds true on GUGP-PWT(1/2) if the 2-to-1 Conjecture holds true. It is shown the  $(1 - \zeta, \delta)$ -gap hardness of problems in GUGP-PWT( $\rho$ ) possesses the compactness property when  $\rho \rightarrow 0$ . The authors pose an open problem whether Unique Game Conjecture holds true on GUGP-PWT( $\rho$ ) with  $0 < \rho < 1$ .

Section 2 demonstrates some definitions and conjectures. The authors show the main results for GUGP-NWA in Section 3 and for GUGP-PWT( $\rho$ ) in Section 4. Section 5 is some discussions.

## 2 Preliminaries

In 2-Prover 1-Round Game Problem, we are given an graph  $G = (V, E)$ , with each edge  $e$  having a weight  $w_e \in \mathbb{Q}^+$ . We are also given a set of  $k$  labels, which we identify with  $[k] = \{1, \dots, k\}$ . Each edge  $e = (u, v)$  in the graph comes equipped with a relation  $R \subseteq [k] \times [k]$ . The solution of the problem is a labeling  $f : V \rightarrow [k]$  that assigns a label to each vertex of  $G$ . An edge  $e = (u, v)$  is said to be satisfied under  $f$  if  $(f(u), f(v)) \in R$ , else is said to be unsatisfied. The object of the problem is to find a labeling maximizing the total weight of the satisfied edges.

Unique Game Problem (UGP) is a special type of 2-Prover 1-Round Game Problem. In UGP, we are given an graph  $G = (V, E)$ , a weight function  $w_e \in \mathbb{Q}^+$  for  $e \in E$ , and a set of labels,  $[k]$ . Each edge  $e = (u, v)$  in the graph comes equipped with a permutation  $\pi_e : [k] \rightarrow [k]$ . The solution of the problem is a labeling  $f : V \rightarrow [k]$  that assigns a label to each vertex of  $G$ . An edge  $e = (u, v)$  is said to be satisfied under  $f$  if  $\pi_e(f(u)) = f(v)$ ,

else is said to unsatisfied. Note that we allow  $G$  is a graph with parallel edges, i.e., there exist more than one edges between two vertices.

It is possible to define two optimization problems in this situation. In Max UGP, the value of the instance is defined as the maximum fraction of the total weight of the satisfied edges. In Min UGP, the value of the instance is defined as the minimum fraction of the total weight of the unsatisfied edges. The total weight of all edges is normalized to 1.

Khot initiates much of the interest in the following conjecture by showing that many hardness results stem from it. It basically states that it is NP-hard to distinguish whether many or only few edges are satisfied.

**Conjecture 1. ([1] Unique Game Conjecture)** *For every  $\zeta, \delta > 0$ , there is a  $k = k(\zeta, \delta)$  such that given an instance  $G$  of Max UGP with  $k$  labels it is NP-hard to distinguish whether  $Val(G) > 1 - \zeta$  or  $Val(G) < \delta$ .*

The conjecture can be restated in Min UGP form, and the two conjectures are equivalent.

**Conjecture 2. (Unique Game Conjecture in Min UGP Form)** *For every  $\zeta, \delta > 0$ , there is a  $k = k(\zeta, \delta)$  such that given an instance  $G$  of Min UGP with  $k$  labels it is NP-hard to distinguish whether  $Val(G) < \zeta$  or  $Val(G) > 1 - \delta$ .*

In 2-Prover 1-Round Game, we are given a graph  $G = (V, E)$ . We are also a set of labels  $[k]$ , where  $[k] = \{1, \dots, k\}$ . Each edge  $e = (u, v)$  in the graph comes equipped with a relation  $R \subseteq [k] \times [k]$ . The labeling of the problem is a function  $f : V \rightarrow [k]$  that assigns a label to each vertex of  $G$ . An edge  $e = (u, v)$  is said to be satisfied under  $f$  if  $(f(u), f(v)) \in R$ , else is said to unsatisfied.

2-to-1 Game and 2-to-2 Game are two special types of 2-Prover 1-Round Game. In 2-to-1 Game Problem, all relations are 2-to-1 projections. A relation  $R \subseteq [k] \times [k]$  is said to be a 2-to-1 projection if there is a map  $\sigma : [k] \rightarrow [k]$  such that for each element  $k \in [k]$  we have  $(i, j) \in R$  iff  $j = \sigma(i)$ , and  $\pi^{-1}(j) \leq 2$ . In 2-to-2 Game Problem, all relations are 2-to-2 relations. A relation  $R \subseteq [2k] \times [2k]$  is said to be a 2-to-2 relation if there are two permutations  $\pi_u, \pi_v : [dk] \rightarrow [2k]$  such that  $(i, j) \in R$  iff  $(\pi_u(i), \pi_v(j)) \in T$  where

$$T := \bigcup_{l=1}^k \{(2l-1, 2l-1), (2l-1, 2l), (2l, 2l-1), (2l, 2l)\}.$$

In this paper, the value of the instance of the two problems,  $Val(G)$ , is defined as the minimum fraction of the unsatisfied edges.

Khot defines the d-to-1 property of 2-Prover 1-Round Game and pose the d-to-1 Conjecture for fixed integer  $d \geq 2$ [1]. Dinur et al. define the 2-to-2 relation and pose the 2-to-2 Conjecture[7] It is easy to prove that the 2-to-1 Conjecture implies the 2-to-2 Conjecture. The author lists the two conjectures in their minimization forms:

**Conjecture 3. (2-to-1 Conjecture)** *For every  $\delta > 0$ , there is a  $k = k(\delta)$  such that given an instance  $G$  of 2-to-1 Game Problem with  $k$  labels it is NP-hard to distinguish whether*

$Val(G) = 0$  or  $Val(G) > 1 - \delta$ .

**Conjecture 4. (2-to-2 Conjecture)** *For every  $\delta > 0$ , there is a  $k = k(\delta)$  such that given an instance of 2-to-2 Game Problem with  $2k$  labels it is NP-hard to distinguish whether  $Val(G) = 0$  or  $Val(G) > 1 - \delta$ .*

In this paper, the author defines Generalized Unique Game Problem (GUGP), where weights of the edges are allowed to be negative. In GUGP, we are given an graph  $G = (V, E)$  possibly having parallel edges, weight function  $w_e \in \mathbb{Q}$  for  $e \in E$ , and the set of labels,  $[k]$ . Each edge  $e = (u, v)$  in the graph comes equipped with a permutation  $\pi_e : [k] \rightarrow [k]$ .

Note that  $w_e$  could be positive or negative. We assume there is no edge with zero weight for sake of clearance. Let  $\Sigma_G$  be the total of the positive weights of edges,  $W_G^+$  be the total of the positive weights of edges, and  $\Sigma_G = W_G^+ + W_G^-$  be the total weight of all edges. We call  $r_G = |W_G^-|/W_G^+$  the negative/positive ratio of the instance.

The solution of GUGP is a labeling  $f : V \rightarrow [k]$  that assigns a label to each vertex of  $G$ . It is also possible to define two optimization problems. In Max GUGP, the goal is to maximize the total weight of the satisfied edges. In Min GUGP, the goal is to minimize the total weight of the unsatisfied edges. GUGP-NWA and GUGP-PWT are two special types of GUGP.

In GUGP-NWA, the weight of all edges are negative. In Max GUGP-NWA, we seek to minimize the total weight of the unsatisfied edges, i.e. to maximize the *absolute value* of the total weight of the unsatisfied edges. The value of Max GUGP-NWA is defined as maximum fraction of the absolute value of the total weight of the unsatisfied edges, in condition  $|W_G^-|$  is normalized to 1. In Min GUGP-NWA, we seek to maximize the total weight of the satisfied edges, i.e. to minimize the *absolute value* of the total weight of the satisfied edges. The value of Min GUGP-NWA is defined as minimum fraction of the absolute value of the total weight of the satisfied edges, in condition  $|W_G^-|$  is normalized to 1.

We give the counterpart of the Unique Game Conjecture on GUGP-NWA as follows:

**Conjecture 5. (Unique Game Conjecture on GUGP-NWA)** *For every  $\zeta, \delta > 0$ , there is a  $k = k(\zeta, \delta)$  such that given an instance of Max GUGP-NWA with  $k$  labels it is NP-hard to distinguish whether  $Val(G) > 1 - \zeta$  or  $Val(G) < \delta$ .*

In GUGP-PWT, the total weight of all edges is positive. In GUGP-PWT, we seek to minimize the total weight of the unsatisfied edges. The value of Min GUGP-PWT is defined as the total weight of the unsatisfied edges divided by  $\Sigma_G$ . Note that the value of GUGP-PWT could be negative or more than 1. In an instance  $G$  of GUGP-PWT, let  $W_G(f)$  be the total weight of the unsatisfied edges under labeling  $f$ , let the optimal labeling be  $f^*$ . The value of the instance is  $Val(G) = W_G(f^*)/\Sigma_G$ . We define GUGP-PWT( $\rho$ ) as the subproblem of GUGP-PWT where the negative/positive ratio is at most  $\rho$ , where  $\rho$  is a constant. Since the negative/positive ratio is always less than 1, we set the range of  $\rho$  to be  $0 < \rho \leq 1$ . Note that GUGP-PWT(0) is Min UGP itself.

We give the counterpart of the Unique Game Conjecture on GUGP-PWT( $\rho$ ) as follows:

**Conjecture 6. (Unique Game Conjecture on GUGP-PWT( $\rho$ ))** *For every  $\zeta, \delta > 0$ , there is a  $k = k(\zeta, \delta)$  such that given an instance of GUGP-PWT( $\rho$ ) with  $k$  labels it is NP-hard to distinguish whether  $\text{Vla}(G) < \zeta$  or  $\text{Val}(G) > 1 - \delta$ .*

### 3 GUGP-NWA

In this section, we prove Min GUGP-NWA is NPO-complete, i.e. it is NP-hard to approximate Min GUGP-NWA within any factor of  $\text{poly}(n)$ , and we prove Max GUGP-NWA can be approximated with factor 2, and Conjecture 5 is refuted as a corollary.

**Theorem 1.** *Min GUGP-NWA is NPO-complete.*

*Proof.* Min GUGP-NWA can be restated as: In the situation of the unique game problem, the goal is to find minimum fraction of the total weight of the *satisfied edges*. We construct an approximation ratio preservation reduction from TSP to the above problem.

Given an instance of TSP problem  $G = (V, E)$ , where each edge of  $E$  has a weight  $w_e \in \mathbb{Q}^+$ . Denote  $n = |V|$ . The instance of Min GUGP-PWT is a graph  $G' = G'(V, E')$ , with each edge  $e' \in E'$  having a weight  $w'(e')$ , and with the labeling set  $[n]$ . For each edge  $e = (u, v) \in E$ , there are three parallel edges  $e^=$ ,  $e^+$  and  $e^-$  between  $u$  and  $v$  in  $E'$ .  $e^=$  has weight  $M$  and equipped with permutation  $\pi^= = \{(1, 1), (2, 2), \dots, (n, n)\}$ . Let  $M = n \cdot \text{Max}(w)$ , where  $\text{Max}(w)$  is the maximum weight of all edges in  $G$ .  $e^+$  has weight  $w(e)$  and equipped with permutation  $\pi^+ = \{(1, 2), (2, 3), \dots, (n, 1)\}$ .  $e^-$  has weight  $w(e)$  and equipped with permutation  $\pi^- = \{(1, n), (2, 1), \dots, (n, n-1)\}$ .

Given a solution of TSP problem, a Hamiltonian cycle  $C$ , we can assign label 1 to  $n$  to vertices of  $C$  along  $C$  in  $G'$ , and the total weight of satisfied edges in  $G'$  is exactly the total weight of edges on  $C$  in  $G$ .

In the other direction, given a labeling  $f$  of  $G'$ , if there are two vertices assigned with the same label, the total weight of the satisfied edges is at least  $M$ . Otherwise all vertices are assigned with label from 1 to  $n$  respectively, let  $u_i$  be the vertices assigned label  $i$  for  $1 \leq i \leq n$ , and  $e_i^+ \in E'$  be the edge between  $u_i$  and  $u_{i+1 \bmod n}$  equipped with permutation  $\pi^+$ . The total weight of the satisfied edges is equal to  $\sum_{1 \leq i \leq n} w'(e_i^+)$ . Let  $C$  be the Hamiltonian cycle of  $G$  which consists of vertices from  $u_1$  to  $u_n$ , then the total weight of  $C$  in  $G$  is exactly the total weight of satisfied edges under  $f$  in  $G'$ .  $\square$

**Theorem 2.** *Max GUGP-NWA can be approximated within factor 2.*

*Proof.* Max GUGP-NWA can be restated as the following 2-prover 1-round game problem. We are given an graph  $G = (V, E)$ , a weight function  $w_e \in \mathbb{Q}^+$ , and the set of labels,  $[k]$ . Each edge  $e = (u, v)$  in the graph comes equipped with a relation  $\bar{\pi}_e = [k] \times [k] - \pi_e$ , where  $\pi_e : [k] \rightarrow [k]$  is a permutation. The solution of the problem is a labeling  $f : V \rightarrow [k]$  that assigns a label to each vertex of  $G$ . An edge  $e = (u, v)$  is said to be satisfied under  $f$  if  $(f(u), f(v)) \in \bar{\pi}_e$ . The value of the instance is defined as the maximum fraction of the total weight of the satisfied edges



We describe an approximation algorithm that finds a solution under which the fraction of the total weight of the satisfied edges is at least  $1/2$ , which is at least half of the value of the instance.

In the beginning of the algorithm, assign arbitrary labels to all vertices. Let  $Ass(v, e)$  be the predicate whether  $v \in V$  is associated with  $e \in E$ , and  $Sat(e)$  be the predicate whether  $e$  is satisfied by current labeling. In each iteration of the algorithm, let  $U^< = \{v \in V \mid \sum_{Ass(v,e) \wedge Sat(e)} w_e < \frac{1}{2} \sum_{Ass(v,e)} w_e\}$ , and  $U^{\geq} = \{v \in V \mid \sum_{Ass(v,e) \wedge Sat(e)} w_e \geq \frac{1}{2} \sum_{Ass(v,e)} w_e\}$ . If  $U^< = \emptyset$ , the algorithm stops. Otherwise, choose a vertex  $u$  from  $U^<$ , suppose the label assigned to  $u$  is  $f_1$ , assign another label  $f_2$  to the vertex. If an edge  $e = (u, v)$  is unsatisfied under the old labeling, it must be the case  $(f_1, f(v)) \notin \pi_e$ , and  $\pi_e(f_1) = f(v)$ . So  $\pi_e(f_2) \neq f(v)$ , and  $(f_2, f(v)) \in \pi_e$ . Therefore, in the new labeling vertex  $u$  satisfies the condition of  $U^{\geq}$ , and we move it from  $U^<$  to  $U^{\geq}$ . Since after each iteration, the number of vertices in  $W$  is increasing by 1, the algorithm stops in  $|V|$  iterations.  $\square$

**Corollary 1.** *Conjecture 5 holds false.*

## 4 GUGP-PWT( $\rho$ )

### 4.1 Parallel Repetition of Max 3-Cut

In Max 3-Cut Problem, the instance is a graph  $G = (V, E)$ , the value of the instance is the maximum fraction of properly colored edges of  $G$  under a 3-coloring. 3-coloring of a graph in a color set  $[3]$  is a function  $\chi : V \rightarrow [3]$ . We say an edge is properly colored under a 3-coloring if its endpoints receive distinct colors. A graph is 3-colorable if there is a coloring under which all edges are proper colored. Max 3-Cut Problem can be viewed as a 2-prover 1-round game with  $k = 3$ , where each edge comes equipped with a relation  $\pi_e = \{(1, 2), (2, 3), (3, 1), (2, 1), (3, 2), (1, 3)\}$ .

Given an instance of Max 3-Cut Problem,  $G$ , we define the  $l$ -fold parallel repetition of the instance,  $G^l = G^l(V^l, E^l)$ , as follows.  $G^l = \{< u_1, \dots, u_l > \mid u_i \in V, 1 \leq i \leq l\}$ , and  $E^l = \{(< u_1, \dots, u_l >, < v_1, \dots, v_l >) \mid (u_i, v_i) \in E, 1 \leq i \leq l\}$ . The value of the instance is the maximum fraction of properly colored edges under a  $l$ -fold 3-coloring. Define a  $l$ -fold 3-coloring of  $G^l$  in the color set  $[3]^l$  be the function  $\chi^l : V \rightarrow [3]^l$ , and let  $m(\chi^l)$  be the number of properly colored edges under  $\chi^l$ . We say an edge  $e = (< u_1, \dots, u_l >, < v_1, \dots, v_l >)$  in  $E^l$  is properly colored under a  $l$ -fold 3-coloring if  $\chi(u_i) \neq \chi(v_i)$  for any  $1 \leq i \leq l$ . The graph  $G^l$  is 3-colorable if there is a  $l$ -fold 3-coloring under which all edges are proper colored.

Petrant [9] shows that Max 3-Cut Problem possesses a hard gap at location 1, i.e. it is NP-hard to distinguish whether the instance is 3-colorable or whether has value at most  $1 - \gamma$  for some constant  $\gamma$ . The constant is presumably very small and not determined in his paper. V. Guruswami et al. [8] make the constant clear to be  $1/33 - \alpha$  for any  $\alpha > 0$ .

**Lemma 1.** ([9] Theorem 3.3) *It is NP-hard to distinguish whether the instance of Max 3-Cut Problem is whether 3-colorable or has value at most  $1 - \gamma$  for some constant  $\gamma > 0$ .*

Raz's Parallel Repetition Theorem is used to enlarge the gap of 2-prover 1-round game with perfect completeness. We introduce Lemma 2 when applying Parallel Repetition Theorem to  $l$ -fold parallel repetition of Max 3-Cut Problem, and get Lemma 3 by a gap-reduction.

**Lemma 2.** ([10] Theorem 1.1) *If an instance of Max 3-Cut Problem has value  $1 - \gamma$ , the value of the  $l$ -fold parallel repetition of the instance is at most  $(1 - \gamma^{c_1})^{c_2^l}$ , where  $c_1$  and  $c_2$  are two positive constants.*

**Lemma 3.** *For any constant  $\delta > 0$ , there is a constant  $l = l(\delta)$  such that it is NP-hard to distinguish whether the instance of  $l$ -fold parallel repetition of Max 3-Cut Problem is  $l$ -fold 3-colorable or has value at most  $\delta$ .*

*Proof.* Given an instance of Max 3-Cut Problem,  $G = (V, E)$ , let  $l$  be the integer no less than  $\frac{\ln \delta}{c_2 \ln(1 - \gamma^{c_1})}$ . Let  $G^l$  be the  $l$ -fold parallel repetition of  $G$ . The proof can be achieved by the following two steps and Lemma 2.

**Completeness.** Suppose  $G$  is 3-colorable. Define a  $l$ -fold 3-coloring of  $G^l$  as  $\chi^l(< v_1, \dots, v_l >) = < \chi^*(v_1), \dots, \chi^*(v_l) >$ , where  $\chi^*$  is the optimal 3-coloring of  $G$ . Since all edges in  $G$  are properly colored under  $\chi^*$ , all edges in  $G^l$  are properly colored under  $\chi^l$ . Therefore,  $G^l$  is  $l$ -fold 3-colorable.

**Soundness.** Suppose  $G$  has value at most  $1 - \gamma$ . By Lemma 2, the value of  $G^l$  is at most  $(1 - \gamma^{c_1})^{c_2^l}$ , which is at most  $\delta$  by the definition of  $l$ .  $\square$

## 4.2 Unique Game Conjecture on GUGP-PWT( $\rho$ )

Let us show Conjecture 6 holds true at the boundary of the range of  $\rho$ .

**Theorem 3.** *Conjecture 6 holds true for  $\rho = 1$ .*

*Proof.* Given two constants  $\zeta, \delta > 0$ , let  $G^l = (V^l, E^l)$  be an instance of the parallel repetition of the Max 3-Cut Problem. The instance of Min GUGP-PWT(1) is a graph  $G' = (V^l, E')$ , with labeling set  $[3^l]$ .

To accomplish the reduction, we design a gadget as replacing each edge  $e = (u, v)$  in  $E^l$  with  $3^l$  parallel edges  $e_{i_1, \dots, i_l}$  for  $1 \leq i_j \leq 3, 1 \leq j \leq l$  between  $u$  and  $v$  in  $E'$ . Let  $E^=$  be the set of edges such that at least one index is 1, and  $E^\neq$  be the set of edges such that all indexes are 2 or 3. Note that  $|E^=| = 3^l - 2^l$  and  $|E^\neq| = 2^l$ . Edges in  $E^=$  has weight  $w_x$ , and edges in  $E^\neq$  has weight  $w_y$ .  $e_{i_1, \dots, i_l}$  for  $1 \leq i_j \leq 3, 1 \leq j \leq l$  is equipped with permutation  $\pi_{i_1, \dots, i_l} = \{(< f_1, \dots, f_l >, < f_1 + i_1 - 1 \bmod 3, \dots, f_l + i_l - 1 \bmod 3 >)\} | f_j \in [3], 1 \leq j \leq l\}$ .

Note that there is always exactly one satisfied edge in  $E'$  between  $u$  and  $v$  under any labeling of  $G'$ . Suppose two vertices  $u$  and  $v$  are assigned labels  $< f_{u,1}, \dots, f_{u,l} >$  and  $< f_{v,1}, \dots, f_{v,l} >$  respectively. We require: (i) when  $f_{u,j} = f_{v,j}$  for at least one  $1 \leq j \leq l$ , the total weight of the unsatisfied edges between  $u$  and  $v$  in  $E'$  is 1; (ii) when  $f_{u,j} \neq f_{v,j}$  for any  $1 \leq j \leq l$ , the total weight of the unsatisfied edges between  $u$  and  $v$  in  $E'$  is 0.

Let us determine the value of  $w_x$  and  $w_y$ . By the linear equations,

$$\begin{cases} (3^l - 2^l - 1)w_x + 2^l w_y = 1 \\ (3^l - 2^l)w_x + (2^l - 1)w_y = 0 \end{cases},$$

we have

$$\begin{cases} w_x = -\frac{2^l - 1}{3^l - 1} \\ w_y = \frac{3^l - 2^l}{3^l - 1} \end{cases}.$$

Note that  $w_x < 0$  and  $w_y > 0$ .  $W_{G'}^+ = \frac{2^l(3^l - 2^l)}{3^l - 1}|E^l|$ ,  $W_{G'}^- = -\frac{(2^l - 1)(3^l - 2^l)}{3^l - 1}|E^l|$ ,  $\Sigma_{G'} = \frac{3^l - 2^l}{3^l - 1}|E^l|$ , and  $r_{G'} = |W_{G'}^-|/W_{G'}^+ = 1 - \frac{1}{2^l} = 1 - O(\delta^c)$ , where  $c$  is a positive constant.

**Completeness.** Suppose  $G^l$  is  $l$ -fold 3-colorable. Let  $\chi^l$  be the optimal  $l$ -fold 3-coloring, then  $m(\chi^l) = |E^l|$ . Let  $f = \chi^l$ . For any edge  $e = (u, v)$  in  $E^l$ ,  $f_{u,j} \neq f_{v,j}$  for any  $1 \leq j \leq l$ , since  $\chi^l$  is a  $l$ -fold 3-coloring. So the total weight of the unsatisfied edges between  $u$  and  $v$  is 0. Therefore, the total weight of the unsatisfied edges in  $E'$  is 0, i.e.  $Val(G') = 0 < \zeta$ .

**Soundness.** Suppose the value of  $G^l$  is at most  $\delta$ . For any labeling  $f$  of  $G'$ ,  $\chi^l = f$  is a  $l$ -fold 3-coloring of  $G^l$ , and  $m(\chi^l) < \delta$ . So at least  $1 - \delta$  fraction of edges in  $E^l$  are not properly  $l$ -fold 3-colored. The two vertices of these edges share the same color in at least one element, in another word, the labels of the two vertices under  $f$  share the same value in at least one element. Since the total weight of the unsatisfied edges between such two vertices in  $E'$  is 1, the total weight of the unsatisfied edges in  $E'$  under  $f$  is at least  $(1 - \delta)|E^l|$ . Therefore,  $Val(G') \geq (1 - \delta)|E^l|/\Sigma_{G'} > 1 - \delta$ .  $\square$

We prove that Conjecture 6 holds true for  $\rho = 1/2$  if Conjecture 4 holds true. Since Conjecture 3 implies Conjecture 4, Conjecture 6 holds true for  $\rho = 1/2$  if Conjecture 3 holds true.

For any two positive integers  $m$  and  $n$ , let  $r$  be the remainder when dividing  $m$  by  $n$ , then  $0 \leq r \leq n - 1$ . Let  $m_{mod\ n} = r$ , if  $r > 0$ ;  $n$ , if  $r = 0$ .

**Theorem 4.** *Conjecture 6 holds true for  $\rho = 1/2$  if Conjecture 4 holds true.*

*Proof.* Given  $\delta > 0$ , let  $G = (V, E)$  be an instance of 2-to-2 Game Problem, with labeling set  $[2k]$ . We construct an instance of Min GUGP-PWT(1/2) as a graph  $G' = (V, E')$ , with labeling set  $[2k]$ . For each edge  $e = (u, v)$  in  $E$  with the 2-to-2 relation  $R$ , let the two permutations w.r.t.  $R$  are  $\pi_u, \pi_v$ , we design a gadget as replacing  $e$  with  $2k$  parallel edges  $e_1, \dots, e_{2k}$  between  $u$  and  $v$  in  $E'$ .

The edge  $e_1$  has weight  $w_x$  and is equipped with the permutation

$$\pi_1 = \{(\pi_u^{-1}(1), \pi_v^{-1}(1)), (\pi_u^{-1}(2), \pi_v^{-1}(2)), \dots, (\pi_u^{-1}(2k-1), \pi_v^{-1}(2k-1)), (\pi_u^{-1}(2k), \pi_v^{-1}(2k))\}.$$

The edge  $e_2$  has weight  $w_x$  and is equipped with the permutation

$$\pi_2 = \{(\pi_u^{-1}(1), \pi_v^{-1}(2)), (\pi_u^{-1}(2), \pi_v^{-1}(1)), \dots, (\pi_u^{-1}(2k-1), \pi_v^{-1}(2k)), (\pi_u^{-1}(2k), \pi_v^{-1}(2k-1))\}.$$

The edge  $e_{2j-1}$  for  $2 \leq j \leq k$  has weight  $w_y$  and is equipped with the permutation

$$\pi_{2j-1} = \bigcup_{i=1}^k \{(\pi_u^{-1}(2i-1), \pi_v^{-1}(2i-1+2j-2_{\text{mod } 2k})), (\pi_u^{-1}(2i), \pi_v^{-1}(2i+2j-2_{\text{mod } 2k}))\}.$$

The edge  $e_{2j}$  for  $2 \leq j \leq k$  has weight  $w_y$  and is equipped with the permutation

$$\pi_{2j} = \bigcup_{i=1}^k \{(\pi_u^{-1}(2i-1), \pi_v^{-1}(2i-1+2j-1_{\text{mod } 2k})), (\pi_u^{-1}(2i), \pi_v^{-1}(2i+2j-3_{\text{mod } 2k}))\}.$$

Note that there is always exactly one satisfied edge in  $E'$  between  $u$  and  $v$  under any labeling of  $G'$ . Suppose two vertices  $u$  and  $v$  are assigned labels  $f_u$  and  $f_v$  respectively. We require: (i) when the edge  $e = (u, v)$  is satisfied under  $f$  in  $G$ , i.e. one of the two edges  $e_1$  and  $e_2$  is satisfied, the total weight of the unsatisfied edges between  $u$  and  $v$  in  $E'$  is 1; (ii) when the edge  $e = (u, v)$  is unsatisfied under  $f$  in  $G$ , i.e. one of the edges  $e_{2j-1}$  and  $e_{2j}$  for  $2 \leq j \leq k$  is satisfied, the total weight of the unsatisfied edges between  $u$  and  $v$  in  $E'$  is 0.

Let us determine the value of  $w_x$  and  $w_y$ . By the linear equations

$$\begin{cases} w_x + (2k-2)w_y = 0 \\ 2w_x + (2k-3)w_y = 1 \end{cases},$$

we have

$$\begin{cases} w_x = \frac{2k-2}{2k-1} \\ w_y = -\frac{1}{2k-1} \end{cases},$$

Note that  $w_x > 0$ ,  $w_y < 0$ ,  $\Sigma_{G'} = \frac{2k-2}{2k-1}|E|$  and  $r_{G'} = 1/2$ .

The proof is completed by the following two steps.

**Completeness.** Suppose  $Val(G) = 0$ . Let  $f$  be the optimal labeling of  $G$ , then  $f$  is also a labeling of  $G'$ . Since any edge  $e = (u, v)$  in  $E$  is satisfied under  $f$ , the total weight of the unsatisfied edges between  $u$  and  $v$  in  $E'$  is 0. Therefore, the total weight of the unsatisfied edges in  $E'$  is 0, i.e.  $Val(G') = 0$ .

**Soundness.** Suppose  $Val(G) > 1 - \delta$ . Then for any labeling  $f$  of  $G'$ ,  $f$  is a labeling of  $G$ , at least  $1 - \delta$  fraction of the edges in  $E$  are unsatisfied. By the definition of  $E'$ , the total weight of the unsatisfied edges in  $E'$  between the two endpoints of such edges is 1. So the total weight of the unsatisfied edges in  $E'$  under  $f$  is at least  $(1 - \delta)|E|$ . Therefore,  $Val(G') \geq (1 - \delta)|E|/\Sigma_{G'} > 1 - \delta$ .  $\square$

In the end of this section, we establish a connection from Conjecture 6 to the Unique Game Conjecture by the following theorem.

**Theorem 5.** *If Conjecture 6 holds true for any  $\rho > 0$ , Conjecture 2 holds true.*

*Proof.* Suppose Conjecture 2 holds false, then for some  $\zeta, \delta > 0$ , for any label size  $k$ , we can decide in polynomial time whether Min UGP with  $k$  labels has a value more than  $1 - \delta$  or less than  $\zeta$ . We claim Conjecture 6 for  $\rho = \min(\zeta, \delta)/2$  holds false.

Given an instance  $G = (V, E)$  of Min GUGP-PWT( $\rho$ ), we construct an instance  $G' = (V, E')$  of Min UGP as follows. Let  $E'$  be the set of the edges in  $E$  with positive weights. Let  $f^*$  be the optimal labeling of  $G$ , and  $f'$  be the optimal labeling of  $G'$ . Then  $Val(G') = W_{G'}(f')/W_G^+$  and  $Val(G) = W_G(f^*)/\Sigma_G$ .

Since  $W_{G'}(f') \geq W_G(f') \geq W_G(f^*)$  and  $\Sigma_G/W_G^+ \geq 1 - \rho$ ,  $Val(G') \geq (1 - \rho)Val(G)$ .

By the definition of  $E'$ ,  $W_G(f^*) \geq W_{G'}(f^*) - \rho W_G^+$ . We have  $Val(G)\Sigma_G = W_G(f^*) \geq W_{G'}(f^*) - \rho W_G^+ \geq W_{G'}(f') - \rho W_G^+ = (Val(G') - \rho)W_G^+$ . Therefore,  $Val(G') \leq Val(G) + \rho$ .

If  $Val(G) < \zeta/2$ , then  $Val(G') < \zeta$ . If  $Val(G) > 1 - \delta/2$ , then  $Val(G') > 1 - \delta$ . Thus we can decide in polynomial time whether the instance of Min GUGP-PWT( $\rho$ ) has a value more than  $1 - \delta/2$  or less than  $\zeta/2$ .  $\square$

## 5 Discussions

We notice the proof of Theorem 3 does not apply to the case for any  $\rho < 1$ , since  $r_{G'} \rightarrow 1$  when  $\delta \rightarrow 0$ . We leave it an open problem whether Unique Game Conjecture holds true on GUGP-PWT( $\rho$ ) for  $0 < \rho < 1$ .

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